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On the Derivation of Quantum Kinetic Equations. II. Nonlocal Uehling–Uhlenbeck Equation

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The binary and triple collision terms of the quantum kinetic equation derived previously are analyzed in the weak coupling approximation. In this approximation the equation appears to be a nonlocal Markovian extension of the kinetic equation due to Uehling and Uhlenbeck. After linearization, its relationship with non-Markovian formulations found in the literature is studied.

KEY WORDS: Nonequilibrium statistical mechanics; quantum kinetic equation; weak coupling approximation; nonlocal collision kernel; memory kernel; Uehling–Uhlenbeck equation.

1. INTRODUCTION

In the preceding paper,[1] hereafter referred to as I, we have established a quantum kinetic equation that takes binary as well as triple collisions into account. In contrast to the classical equation,[2] where the collision kernels are operators in phase space, the quantum mechanical kernels are ordinary functions. This is a definite advantage, which makes a study of the detailed analytic properties by perturbation methods a feasibility. In the present paper we make a start with this program and evaluate the two- and three-particle collision terms up to second order in the interparticle potential.

Weak coupling approximations have been considered by many authors, using a multitude of methods, such as truncation of the hierarchy,[3,4] master equation approach,[5] Green's function techniques,[6–9] projector formalism,[10] and others.[11,12] A separate class consists of the theories[13,14] that are

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a priori restricted to the linear response regime. We make contact with these by linearizing with respect to the equilibrium state.

In a number of the aforementioned theories[5–8,13] the kinetic equation appears in a non-Markovian form, that is, the collision term is a time integral extending over the whole past history of the system. In our treatment, on the other hand, the kinetic equation is strictly Markovian. To resolve this apparent inconsistency, we note that in the weak coupling case the distribution function in the second-order collision terms may be treated as if they had a free time evolution. It is then a simple matter to convert our Markovian equation into a non-Markovian form.

The preceding argument implies that a non-Markovian collision term does not necessarily represent a memory effect. The deciding factor for that is the choice of the initial time. It is customary to situate it at time zero. One then finds a collision kernel containing a memory effect that vanishes for long times. In contrast, we impose the initial condition in the infinite past. As a consequence, the ensuing kinetic equation is Markovian at all times. Apparently, the initial condition at minus infinity automatically takes care of the asymptotic time limit.

We examine further the relationship between the Markovian and memory kernel by making a Fourier–Laplace transform of the linearized equation. This reveals that the Markovian kernel, which is only wave-vector-dependent, is not, as is sometimes stated,[15] the zero-frequency limit of the memory kernel. Actually, we find a dispersion relation which, in the weak coupling approximation, has a particularly simple form. No restriction on the wave-vector dependence is involved. This serves to show that in our scheme the Markovian limit and the hydrodynamic limit of small wave vectors are entirely disjunct operations.

2. PERTURBATION EXPANSION

We recall that in paper I, formulas (11), (13), and (24), we have defined functions $\Phi_{\text{in}}^{(n)}(\mathbf{p}^n; \mathbf{p}'^n|\mathbf{p})$ by expanding the operator

$$\Phi(\mathbf{p}) = (i\hbar)^{-1}(2\pi\hbar)^{-3}\int d^3u\,[a^\dagger(\mathbf{p} - \tfrac{1}{2}\mathbf{u})a(\mathbf{p} + \tfrac{1}{2}u), H_I] \tag{1}$$

where H_I is the interaction Hamiltonian, with respect to normal ordered products of in-operators:

$$\Phi(\mathbf{p}) = \sum_{n=2}^{\infty}\frac{1}{n!}\int d^3p^n\, d^3p'^n\, \Phi_{\text{in}}^{(n)}(\mathbf{p}^n; \mathbf{p}'^n|\mathbf{p})a_{\text{in}}^\dagger(\mathbf{p}^n)a_{\text{in}}(\mathbf{p}'^n) \tag{2}$$

We have used the short-hand notations $\mathbf{p}^n = \mathbf{p}_1,\ldots,\mathbf{p}_n$; $d^3p^n = d^3p_1 \cdots d^3p_n$ and $a_{\text{in}}^\dagger(\mathbf{p}^n) = a_{\text{in}}^\dagger(\mathbf{p}_1)\cdots a_{\text{in}}^\dagger(\mathbf{p}_n)$. The in-operators were defined as

$$a_{\text{in}}(\mathbf{p}) = U(0, -\infty)a(\mathbf{p})U^\dagger(0, -\infty) \tag{3}$$

with $U(0, -\infty)$ the Møller wave operator (I.16). Because we assume $U(0, -\infty)$ to be unitary, they satisfy the (anti-) commutation relation

$$a_{\text{in}}(\mathbf{p})a_{\text{in}}^{\dagger}(\mathbf{p}') - \eta a_{\text{in}}^{\dagger}(\mathbf{p}')a_{\text{in}}(\mathbf{p}) = \delta^{(3)}(\mathbf{p} - \mathbf{p}') \tag{4}$$

where η is equal to one for bosons and to minus one for fermions.

In the kinetic theory as developed in paper I the dynamical functions $\Phi_{\text{in}}^{(n)}$ are of prime importance since they effectively determine the collision term of the kinetic equation. They can be found by a systematic procedure based on perturbation theory. To that end we consider the quantity

$$\mathscr{U}^{\dagger}(t, t_0)\Phi(\mathbf{p}) = U^{\dagger}(t, t_0)\Phi(\mathbf{p})U(t, t_0) \tag{5}$$

where $\mathscr{U}^{\dagger}(t, t_0)$ is the superoperator defined by the right-hand side. The time evolution operator $U(t, t_0)$ satisfies the integral equation (I.14), whence

$$i\hbar\, \partial_t \mathscr{U}^{\dagger}(t, t_0) = \mathscr{U}^{\dagger}(t, t_0)\hat{\mathscr{H}}_I(t) \tag{6}$$

where the superoperator $\hat{\mathscr{H}}_I(t)$ indicates the commutator with the interaction Hamiltonian in the interaction picture $\hat{H}_I(t)$. The integral form of (6) is

$$\mathscr{U}^{\dagger}(t, t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, e^{-\epsilon|t_1|}\mathscr{U}^{\dagger}(t_1, t_0)\hat{\mathscr{H}}_I(t_1) \tag{7}$$

By iteration we get the perturbation expansion[16]

$$\mathscr{U}^{\dagger}(t, t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} dt_n\, \hat{\mathscr{H}}_I(t_n) \cdots \hat{\mathscr{H}}_I(t_1) \tag{8}$$

where the convergence factors have been suppressed.

We now write the identity

$$\Phi(\mathbf{p}) = \mathscr{U}(0, -\infty)\mathscr{U}^{\dagger}(0, -\infty)\Phi(\mathbf{p}) \tag{9}$$

Observing that in terms of $\mathscr{U}$ the definition (3) of the in-operators reads

$$a_{\text{in}}(\mathbf{p}) = \mathscr{U}(0, -\infty)a(\mathbf{p}) \tag{10}$$

we find with (8) and (9) the following perturbation series for the operator $\Phi(\mathbf{p})$:

$$\Phi(\mathbf{p}) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_{n-1}} dt_n \times [[\cdots[\Phi_{\text{in}}(\mathbf{p}), H_{I,\text{in}}(t_1)]\cdots], H_{I,\text{in}}(t_n)] \tag{11}$$

The operators $H_{I,\text{in}}(t)$ and $\Phi_{\text{in}}(\mathbf{p})$ are identical to the corresponding operators $H_I(t)$ and $\Phi(\mathbf{p})$ as defined in (I.3) and (1), respectively, save for the replacement of all a-operators by a_{in}-operators. The latter follow the free time development law as shown in (I.22). With the help of (11) the expansion functions $\Phi_{\text{in}}^{(n)}$ as

defined by the expansion (2) can be calculated up to any desired order of approximation.

3. FIRST-ORDER APPROXIMATION

Let us consider the first-order approximation as it follows from (11):

$$\Phi(\mathbf{p}) = \Phi_{\text{in}}(\mathbf{p}) - (i/\hbar)\int_{-\infty}^{0} dt_1\, e^{\epsilon t_1}[\Phi_{\text{in}}(\mathbf{p}), H_{I,\text{in}}(t_1)] \tag{12}$$

The calculation of the right-hand side is a straightforward exercise. Putting the in-construction operators in their normal order as in (2), we may identify the lowest order contributions to the functions $\Phi_{\text{in}}^{(n)}$. As it turns out, we have a first-order contribution to $\Phi_{\text{in}}^{(2)}$, which we denote by $\Phi_{\text{in}}^{(2,1)}$. It reads

$$\begin{aligned}\Phi_{\text{in}}^{(2,1)}&(\mathbf{p}^2; \mathbf{p}'^2|\mathbf{p}) \\ &= (i\lambda/\pi^3\hbar^4)[\delta^{(3)}(2\mathbf{p} - \mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)v(\mathbf{p}^2; 2\mathbf{p} - \mathbf{p}_1', \mathbf{p}_2') \\ &\quad - \delta^{(3)}(2\mathbf{p} - \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')v(2\mathbf{p} - \mathbf{p}_1, \mathbf{p}_2; \mathbf{p}'^2) + (1 \leftrightarrow 2)]\end{aligned} \tag{13}$$

The symbol $(1 \leftrightarrow 2)$ indicates the preceding terms with the labels 1 and 2 interchanged. The potential function v, which characterizes the interaction, is by definition a symmetric or antisymmetric function of both its first and second pair of variables.

The calculation also yields a second order contribution

$$\begin{aligned}\Phi_{\text{in}}^{(2,2)}(\mathbf{p}^2; \mathbf{p}'^2|\mathbf{p}) = \int d^3q^2\, &[U_{\mp}^{(1)}(\mathbf{p}^2; \mathbf{q}^2)\Phi_{\text{in}}^{(2,1)}(\mathbf{q}^2; \mathbf{p}'^2|\mathbf{p}) \\ &- U_{\mp}^{(1)}(\mathbf{q}^2; \mathbf{p}'^2)\Phi_{\text{in}}^{(2,1)}(\mathbf{p}^2; \mathbf{q}^2|\mathbf{p})]\end{aligned} \tag{14}$$

where it has been convenient to define a function

$$U_{\mp}^{(1)}(\mathbf{p}^2; \mathbf{p}'^2) = 2\pi i\lambda\, \delta_{\mp}^{(4)}(\mathbf{p}^2; \mathbf{p}'^2)v(\mathbf{p}^2; \mathbf{p}'^2) \tag{15}$$

with the notation

$$2\pi i\, \delta_{\mp}^{(4)}(\mathbf{p}^2; \mathbf{p}'^2) = \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\left[\sum_{j=1}^{2}(p_j^0 - p_j'^0) - i\epsilon\right]^{-1} \tag{16}$$

The energy $\mathbf{p}^2/2m$ has been indicated by p^0.

Finally, we find from the calculation the first nonvanishing contribution to $\Phi_{\text{in}}^{(3)}$

$$\begin{aligned}\Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3; \mathbf{p}'^3|\mathbf{p}) = \eta\sum_{\mathscr{P}}\eta^{\mathscr{P}}\int d^3q\, &[U_{\mp}^{(1)}(\mathbf{p}^2; \mathbf{q}, \mathbf{p}_3')\Phi_{\text{in}}^{(2,1)}(\mathbf{q}, \mathbf{p}_3; \mathbf{p}'^2|\mathbf{p}) \\ &- U_{\mp}^{(1)}(\mathbf{q}, \mathbf{p}_3; \mathbf{p}'^2)\Phi_{\text{in}}^{(2,1)}(\mathbf{p}^2; \mathbf{q}, \mathbf{p}_3'|\mathbf{p})]\end{aligned} \tag{17}$$

The sum in this expression runs over all permutations of the variables $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. In the lowest order approximation there are no contributions to the expansion (2) other than (13), (14), and (17).

In paper I we also considered the operator

$$\psi(\mathbf{p}) = (2\pi\hbar)^{-3} \int d^3u\, a^\dagger(\mathbf{p} - \tfrac{1}{2}\mathbf{u}) a(\mathbf{p} + \tfrac{1}{2}\mathbf{u}) \tag{18}$$

and the expansion to match

$$\psi(\mathbf{p}) - \psi_{\text{in}}(\mathbf{p}) = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^3p^n\, d^3p'^n\, \psi_{\text{in}}^{(n)}(\mathbf{p}^n; \mathbf{p}'^n|\mathbf{p}) a_{\text{in}}^\dagger(\mathbf{p}^n) a_{\text{in}}(\mathbf{p}'^n) \tag{19}$$

The first-order approximation gives rise to one coefficient function only, which is

$$\psi_{\text{in}}^{(2,1)}(\mathbf{p}^2; \mathbf{p}'^2|\mathbf{p}) = \frac{1}{(\pi\hbar)^3} [U^{(1)}(\mathbf{p}^2; 2\mathbf{p} - \mathbf{p}_1', \mathbf{p}_2') - U^{(1)}(2\mathbf{p} - \mathbf{p}_1, \mathbf{p}_2; \mathbf{p}'^2) + (1 \leftrightarrow 2)] \tag{20}$$

In the next section we shall see how the functions (13), (14), (17), and (20) determine the kinetic equation that is valid up to second order of the interaction strength.

4. TWO-PARTICLE COLLISION TERMS

We now turn to the kinetic equation for the distribution (Wigner) function $f(\mathbf{x}, \mathbf{p}, t)$ derived in paper I, Section 5. In the approximation considered above this equation becomes

$$\left(\partial_t + \frac{\mathbf{p}}{m} \cdot \nabla\right) f = J^{(2,1)} + J^{(2,2)} + J^{(3,2)} \tag{21}$$

where the collision term has been divided into three distinct parts that will be discussed separately.

The first two collision terms are quadratic in the distribution function. They may be written as

$$J^{(2,m)} = \int dx^2\, \tilde{\Phi}_{\text{in}}^{(2,m)}(x^2|x) \prod_{j=1}^{2} f(x_j, t), \qquad m = 1, 2 \tag{22}$$

where the phase-space point $(\mathbf{x}, \mathbf{p})$ is denoted by x. The functions $\tilde{\Phi}_{\text{in}}^{(2,m)}$ are Fourier transforms [cf. (I.34)] of the functions $\Phi_{\text{in}}^{(2,1)}$ and $\Phi_{\text{in}}^{(2,2)}$ written out in (13) and (14), respectively. By substitution of the former one finds that $J^{(2,1)}$ is identical to the collision term that is obtained by applying the well-known

Hartree–Fock factorization rule to the quantum mechanical hierarchy.[17] For systems that are spatially uniform the collision term vanishes.

In order to get some insight into the significance of the collision term (22) for $m = 2$, we make the change $\mathbf{x}_j \to \mathbf{x} + \mathbf{x}_j$ in the spatial integration variables and expand the distribution functions in a Taylor series around the point $\mathbf{x}$:

$$f(\mathbf{x} + \mathbf{x}_j, \mathbf{p}_j, t) = f(\mathbf{x}, \mathbf{p}_j, t) + \mathbf{x}_j \cdot \nabla f(\mathbf{x}, \mathbf{p}_j, t) + \cdots \tag{23}$$

As the distribution function f is supposed to describe the macroscopic behavior of the system, we expect the corrections arising from the second and higher order terms to be small. If they are omitted, the collision term $J^{(2,2)}$ reduces to the localized form

$$J_L^{(2,2)} = (2\pi\hbar)^6 \int d^3p^2\, \Phi_{\text{in}}^{(2,2)}(\mathbf{p}^2; \mathbf{p}^2|\mathbf{p}) \prod_{j=1}^{2} f(\mathbf{x}, \mathbf{p}_j, t) \tag{24}$$

where the integral kernel is given by expression (14) with $\mathbf{p}_1'$ and $\mathbf{p}_2'$ put equal to $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. With the help of the formulas (I.4), (13), (15), and (16) and the well-known identity

$$\lim_{\epsilon \to 0} [(a - i\epsilon)^{-1} - (a + i\epsilon)^{-1}] = 2\pi i\, \delta(a) \tag{25}$$

the diagonal part of (14) may be shown to become

$$\Phi_{\text{in}}^{(2,2)}(\mathbf{p}^2; \mathbf{p}^2|\mathbf{p}) = \tfrac{1}{2}(2\pi\hbar)^{-6} \int d^3p'^2\, W^{(2)}(\mathbf{p}^2|\mathbf{p}'^2) \times [\delta^{(3)}(\mathbf{p} - \mathbf{p}_1') - \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) + (1 \to 2)] \tag{26}$$

The symbol $W^{(2)}$ stands for

$$W^{(2)}(\mathbf{p}^2|\mathbf{p}'^2) = 2(2\pi)^4\hbar^2\lambda^2\, \delta^{(4)}(p_1 + p_2 - p_1' - p_2')|v(\mathbf{p}'^2; \mathbf{p}^2)|^2 \tag{27}$$

This expression, wherein $\delta^{(4)}(p)$ indicates $\delta(p^0)\, \delta^{(3)}(\mathbf{p})$, is the quantum mechanical transition rate evaluated in the Born approximation. Inserting (26) into (24), one recognizes the latter as the familiar Boltzmann collision term. In its nonlocal form (22), it takes the duration and spatial extent of the collision process into account.

5. THREE-PARTICLE COLLISION TERM

The third collision term on the right-hand side of (21) involves, in contrast to the ones discussed above, three distribution functions:

$$J^{(3,2)} = \int dx^3\, [\tilde{\Phi}_{\text{in}}^{(3,2)}(x^3|x) - 2 \int dy\, \tilde{\psi}_{\text{in}}^{(2,1)}(x^2|y) \tilde{\Phi}_{\text{in}}^{(2,1)}(y, x_3|x)] \prod_{j=1}^{3} f(x_j, t) \tag{28}$$

It is most easily analyzed by replacing the $\tilde{\Phi}_{\text{in}}$ and $\tilde{\psi}_{\text{in}}$ functions by their momentum-space counterparts Φ_{in} and ψ_{in} [cf. (I.34)]. The three-particle collision term (28) may thus be cast into the form

$$J^{(3,2)} = \int dx^3 \int d^3u^3\, \Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3 - \tfrac{1}{2}\mathbf{u}^3; \mathbf{p}^3 + \tfrac{1}{2}\mathbf{u}^3|\mathbf{p})_C \times \prod_{j=1}^{3} [\exp(-i\mathbf{u}_j \cdot \mathbf{x}_j/\hbar)] f(\mathbf{x} + \mathbf{x}_j, \mathbf{p}_j, t) \tag{29}$$

Here the subscript C has been used to indicate the combination

$$\Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3; \mathbf{p}'^3|\mathbf{p})_C = \Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3; \mathbf{p}'^3|\mathbf{p}) - 2(2\pi\hbar)^3 \int d^3q\, \psi_{\text{in}}^{(2,1)}(\mathbf{p}^2; \mathbf{p}'^2|\mathbf{q}) \times \Phi_{\text{in}}^{(2,1)}(\mathbf{q} + \tfrac{1}{2}\mathbf{v}, \mathbf{p}_3; \mathbf{q} - \tfrac{1}{2}\mathbf{v}, \mathbf{p}_3'|\mathbf{p}) \tag{30}$$

where $\mathbf{v}$ is an abbreviation for $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2'$, and where $\Phi_{\text{in}}^{(2,1)}$, $\Phi_{\text{in}}^{(3,2)}$, and $\psi_{\text{in}}^{(2,1)}$ have been given in (13), (17), and (20), respectively.

We proceed by inserting the expressions (17) and (20). We then find, noting that the function $U_{-}^{(1)}$ as defined in (15) contains a delta function, that the first term at the right-hand side of (30) and the last term are quite similar. In fact, the latter and certain permutation terms of the former cancel. There remains

$$\Phi^{(3,2)}(\mathbf{p}^3; \mathbf{p}'^3|\mathbf{p})_C = 2\eta \int d^3q\, [U_{-}^{(1)}(\mathbf{p}^2; \mathbf{q}, \mathbf{p}_3')\Phi_{\text{in}}^{(2,1)}(\mathbf{q}, \mathbf{p}_3; \mathbf{p}'^2|\mathbf{p}) - U_{-}^{(1)}(\mathbf{q}, \mathbf{p}_3; \mathbf{p}'^2)\Phi_{\text{in}}^{(2,1)}(\mathbf{p}^2; \mathbf{q}, \mathbf{p}_3'|\mathbf{p})] \tag{31}$$

with $\Phi_{\text{in}}^{(2,1)}$ given by (13).

Expression (29), with (31) inserted, is the three-particle collision term evaluated up to order λ^2. In this order it only takes the presence of a third particle into account through the boson or fermion symmetry. This is most clearly seen in the local approximation in which we disregard the $\mathbf{x}_j$ dependences of the distribution functions. The collision term (29) then becomes

$$J_L^{(3,2)} = (2\pi\hbar)^9 \int d^3p^3\, \Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3; \mathbf{p}^3|\mathbf{p})_C \prod_{j=1}^{3} f(\mathbf{x}, \mathbf{p}_j, t) \tag{32}$$

Putting $\mathbf{p}'^3$ equal to $\mathbf{p}^3$ in (31) we find, using (13), (15), (16), and (25),

$$\Phi_{\text{in}}^{(3,2)}(\mathbf{p}^3; \mathbf{p}^3|\mathbf{p})_C = \eta(2\pi\hbar)^{-6} \int d^3p_1'\, W^{(2)}(\mathbf{p}^2|\mathbf{p}_1', \mathbf{p}_3) \times [\delta^{(3)}(\mathbf{p} - \mathbf{p}_1') - \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) + (\mathbf{p}_1', \mathbf{p}_1 \to \mathbf{p}_3, \mathbf{p}_2)] \tag{33}$$

where $W^{(2)}$ is the transition rate (27). If (33) is inserted into (32), the ensuing

expression is found to be identical to the three-particle collision term advanced by Uehling and Uhlenbeck as early as 1933.[18] They obtained it from the Boltzmann collision term by the substitution of statistical factors appropriate to Bose or Fermi statistics. Here it arises as the weak coupling approximation to the quantum mechanical Choh–Uhlenbeck collision term.[1]

6. LINEARIZED EQUATION

The kinetic equation discussed above is nonlinear in the distribution function. This feature makes the study of its properties an unpromising enterprise. If, however, the system is near its equilibrium state, the nonlinear equation may be replaced by an equation of evolution that is linear in the deviation from equilibrium. The latter is obtained by writing

$$f(\mathbf{x}, \mathbf{p}, t) = f_{\mathrm{eq}}(\mathbf{p}) + \tilde{h}(\mathbf{x}, \mathbf{p}, t) \tag{34}$$

where $f_{\mathrm{eq}}(\mathbf{p})$ is the Bose–Einstein or Fermi–Dirac distribution function. We substitute the form (34) into the kinetic equation (21) with the collision terms as given by (22) and (29). Since we suppose the deviation from equilibrium to be small, we retain only the terms linear in $\tilde{h}(\mathbf{x}, \mathbf{p}, t)$. It is convenient to express the result as an equation for the Fourier transform

$$h(\mathbf{k}, \mathbf{p}, t) = (2\pi\hbar)^{-3} \int d^3x \exp(-i\mathbf{k}\cdot\mathbf{x}/\hbar)\, \tilde{h}(\mathbf{x}, \mathbf{p}, t) \tag{35}$$

After some manipulations we obtain the linear kinetic equation

$$(i\hbar\partial_t - \mathbf{p}\cdot\mathbf{k}/m)h(\mathbf{k}, \mathbf{p}, t) = \sum_{n=2}^{3} \sum_{m=n-1}^{2} \int d^3p'\, I^{(n,m)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) h(\mathbf{k}, \mathbf{p}', t) \tag{36}$$

The (time-independent) integral kernels are given by

$$I^{(2,m)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) = 2(2\pi)^6\hbar^7 i \int d^3p_1 f_{\mathrm{eq}}(\mathbf{p}_1) \Phi_{\mathrm{in}}^{(2,m)}(\mathbf{p}' - \tfrac{1}{2}\mathbf{k}, \mathbf{p}_1; \mathbf{p}' + \tfrac{1}{2}\mathbf{k}, \mathbf{p}_1|\mathbf{p}) \tag{37}$$

where m can take the values one and two, and

$$\begin{aligned} I^{(3,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) = (2\pi)^9\hbar^{10} i \int d^3p^2 f_{\mathrm{eq}}(\mathbf{p}_1) f_{\mathrm{eq}}(\mathbf{p}_2) \\ \times [2\Phi_{\mathrm{in}}^{(3,2)}(\mathbf{p}' - \tfrac{1}{2}\mathbf{k}, \mathbf{p}^2; \mathbf{p}' + \tfrac{1}{2}\mathbf{k}, \mathbf{p}^2|\mathbf{p})_C \\ + \Phi_{\mathrm{in}}^{(3,2)}(\mathbf{p}^2, \mathbf{p}' - \tfrac{1}{2}\mathbf{k}; \mathbf{p}^2, \mathbf{p}' + \tfrac{1}{2}\mathbf{k}|\mathbf{p})_C] \end{aligned} \tag{38}$$

Since the deviation function $\tilde{h}(\mathbf{x}, \mathbf{p}, t)$ is real, the integral kernels have the property

$$I^{(n,m)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) = -I^{(n,m)*}(\mathbf{p}'|-\mathbf{k}, \mathbf{p}) \tag{39}$$

The kernels may be expressed in terms of the potential function by inserting the explicit forms (13), (14), and (31). In the simple case that the interaction only depends on the distance between the particles, the potential function may be written as

$$v(\mathbf{p}^2; \mathbf{p}'^2) = \tfrac{1}{2}[v(|\mathbf{p}_1 - \mathbf{p}_1'|) + \eta v(|\mathbf{p}_1 - \mathbf{p}_2'|)] \tag{40}$$

where $v(k)$ is a real function. For an interaction of this kind the kernel (37) reads for $m = 1$

$$\begin{aligned} I^{(2,1)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) = (2\pi\hbar)^3\lambda \int d^3p_1 f_{\text{eq}}(\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k}) \\ \times \{[v(|\mathbf{k}|) + \eta v(|\mathbf{p} - \mathbf{p}'|)]\, \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) \\ - \eta v(|\mathbf{p} - \mathbf{p}_1|)\, \delta^{(3)}(\mathbf{p} - \mathbf{p}')\} - (\mathbf{k} \leftrightarrow -\mathbf{k}) \end{aligned} \tag{41}$$

Obviously, it has the property (39) and vanishes if the system is uniform, i.e., $\mathbf{k} = 0$. The first term not containing the factor η can be interpreted as a mean field contribution.[6] In the classical limit it reduces to the linearized Vlasov collision kernel. The additional terms represent quantum mechanical exchange effects.[13]

For the collision kernel $I^{(2,2)}$ we find in the same way

$$\begin{aligned} I^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) = (2\hbar)^3\pi^4\lambda^2 i \int d^3p^3\, w(\mathbf{p}', \mathbf{p}^3|\mathbf{p}) \\ \times [f_{\text{eq}}(\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k})\, \delta_-^{(4)}(\mathbf{p}_2, \mathbf{p}_3; \mathbf{p}', \mathbf{p}_1 - \mathbf{k})] - (\text{c.c.}, \mathbf{k} \leftrightarrow -\mathbf{k}) \end{aligned} \tag{42}$$

with $\delta_-^{(4)}$ as defined in (16) and with the abbreviation

$$\begin{aligned} w(\mathbf{p}', \mathbf{p}^3|\mathbf{p}) = [v(|\mathbf{p}' - \mathbf{p}_2|) + \eta v(|\mathbf{p}' - \mathbf{p}_3|)] \\ \times \{[v(|\mathbf{p} - \mathbf{p}'|) + \eta v(|\mathbf{p} - \mathbf{p}_1|)][\delta^{(3)}(\mathbf{p} - \mathbf{p}_2) + \eta\, \delta^{(3)}(\mathbf{p} - \mathbf{p}_3)] \\ - (\mathbf{p}_2, \mathbf{p}_3 \leftrightarrow \mathbf{p}', \mathbf{p}_1)\} \end{aligned} \tag{43}$$

It turns out that the collision kernel $I^{(3,2)}$ follows from the expression (42) for $I^{(2,2)}$ by the prescription

$$\begin{aligned} f_{\text{eq}}(\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k}) \\ \to (2\pi\hbar)^3\eta[2f_{\text{eq}}(\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k})f_{\text{eq}}(\mathbf{p}_2 + \tfrac{1}{2}\mathbf{k}) - f_{\text{eq}}(\mathbf{p}_2 + \tfrac{1}{2}\mathbf{k})f_{\text{eq}}(\mathbf{p}_3 + \tfrac{1}{2}\mathbf{k})] \end{aligned} \tag{44}$$

Formula (42) shows that in general the interaction between the particles cannot be described by an energy-conserving transition probability. If, however, $\mathbf{k} = 0$, the $\delta_-^{(4)}$ function inside the square brackets may be combined with its complex conjugate into a delta function that conserves momentum as

well as energy. One may verify that then $I^{(2,2)}$ and $I^{(3,2)}$ reduce to the linearized Boltzmann and Uehling–Uhlenbeck collision kernels, respectively.

7. MEMORY KERNEL

The kinetic equation (36) is clearly Markovian. It can, however, also be brought into a non-Markovian form. To this end we note that, within the weak coupling approximation and with regard to the collision terms that are quadratic in the interparticle potential, the deviation function $h(\mathbf{k}, \mathbf{p}', t)$ may be treated as if it develops freely in time:

$$h(\mathbf{k}, \mathbf{p}', t) = \exp(-i\mathbf{p}'\cdot\mathbf{k}\tau/m\hbar)\, h(\mathbf{k}, \mathbf{p}', t - \tau) \tag{45}$$

Making use of this property, we may write

$$\int d^3p'\, I^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p})h(\mathbf{k}, \mathbf{p}', t) = \int d^3p' \int_0^\infty d\tau\, I^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}, \tau)h(\mathbf{k}, \mathbf{p}, t - \tau) \tag{46}$$

where the time-dependent kernel is equal to the time-independent one (42), save for the replacement

$$\begin{aligned} &2\pi\, \delta^{(4)}(\mathbf{p}_2, \mathbf{p}_3; \mathbf{p}', \mathbf{p}_1 - \mathbf{k}) \\ &\quad \to \hbar^{-1}\, \delta^{(3)}(\mathbf{p}' + \mathbf{p}_1 - \mathbf{k} - \mathbf{p}_2 - \mathbf{p}_3) \\ &\qquad \times \exp\{i[(\mathbf{p}' - \tfrac{1}{2}\mathbf{k})^0 + (\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k})^0 - (\mathbf{p}_2 + \tfrac{1}{2}\mathbf{k})^0 - (\mathbf{p}_3 + \tfrac{1}{2}\mathbf{k})^0 + i\epsilon]\tau/\hbar\} \end{aligned} \tag{47}$$

The three-particle collision term may be transformed in a similar fashion.

The important point to observe about the right-hand side of (46) is that the time integration extends to infinity. This feature has been introduced into the theory by choosing the time at which the initial correlations were neglected as minus infinity. [See, in particular, formulas (I.16), (I.19), (I.28).] The initial condition could alternatively have been imposed at time zero. One then would have found, instead of (46), a convolution integral with the integration extending to time t. In terms of Laplace-transformed quantities $\bar{I}^{(2,2)}$ and $\bar{h}$ we would then have

$$\int d^3p' \int_0^t d\tau\, I^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}, \tau)h(\mathbf{k}, \mathbf{p}, t - \tau) = \mathscr{L}^{-1}\left[\int d^3p'\, \bar{I}^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}, z)\bar{h}(\mathbf{k}, \mathbf{p}, z)\right] \tag{48}$$

where $\mathscr{L}^{-1}$ indicates the inverse Laplace transform. The z-dependent kernel follows from (42) via the substitution

$$2\pi\,\delta^{(4)}_{\pm}(\mathbf{p}_2, \mathbf{p}_3; \mathbf{p}', \mathbf{p}_1 - \mathbf{k}) \to i\,\delta^{(3)}(\mathbf{p}' + \mathbf{p}_1 - \mathbf{k} - \mathbf{p}_2 - \mathbf{p}_3) \times [(\mathbf{p}' - \tfrac{1}{2}\mathbf{k})^0 + (\mathbf{p}_1 - \tfrac{1}{2}\mathbf{k})^0 - (\mathbf{p}_2 + \tfrac{1}{2}\mathbf{k})^0 - (\mathbf{p}_3 + \tfrac{1}{2}\mathbf{k})^0 + z + i\epsilon]^{-1} \tag{49}$$

A memory kernel having this structure has recently been obtained by Boley and Smith.[13] They surmise that this memory kernel furnishes a description of the short-time behavior of systems that are carefully prepared in accordance with the imposed initial condition. The asymptotic initial condition, on the other hand, applies to any system, whatever its initial state, provided that the correlations decay more rapidly in time than the one-particle distribution function. The restriction lies in the fact that the ensuing kinetic equation pertains to the long-time behavior of the system.

In the long-time limit the two alternative initial conditions should yield the same results. The associated value of z is found by observing that, due to the definition (16) of $\delta^{(4)}_{\pm}$, the left- and the right-hand sides of (49) become identical for the value $z = \mathbf{p}'\cdot\mathbf{k}/m$. We thus have the equality

$$\bar{I}^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}, z = \mathbf{p}'\cdot\mathbf{k}/m) = I^{(2,2)}(\mathbf{p}'|\mathbf{k}, \mathbf{p}) \tag{50}$$

between the memory and Markovian kernels.

It should be noted that no assumption about the wave-vector dependence has been made. If, in addition, the wave vector is required to be small, the right-hand side reduces to the linearized Boltzmann collision kernel, as already mentioned.

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